

# The critical group of $C_4 \times C_n$ \*

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## ABSTRACT

In this paper, the critical group structure of the Cartesian product graph  $C_4 \times C_n$  is determined, where  $n \geq 3$ .

**Keywords** Graph; Laplacian matrix; Critical group; Invariant factor; Smith normal form; Tree number.

**1991 AMS subject classification:** 15A18, 05C50

## 1 Introduction

Let  $G = (V, E)$  be a finite connected graph without self-loops, but with multiple edges allowed. Then the Laplacian matrix of  $G$  is the  $|V| \times |V|$  matrix defined by

$$L(G)_{uv} = \begin{cases} d(u), & \text{if } u = v, \\ -a_{uv}, & \text{if } u \neq v, \end{cases} \quad (1.1)$$

where  $a_{uv}$  is the number of the edges joining  $u$  and  $v$ , and  $d(u)$  is the degree of  $u$ .

Regarding  $L(G)$  as representing an abelian group homomorphism:  $Z^{|V|} \rightarrow Z^{|V|}$ , its cokernel  $\text{coker}(L(G)) = Z^{|V|}/\text{im}(L(G))$  is an abelian group, determined by the generators  $x_1, \dots, x_{|V|}$  and relation  $L(G)X = 0$ , where  $x_i = (0, \dots, 0, 1, 0, \dots, 0) \in Z^{|V|}$ , whose unique nonzero 1 is in position  $i$ , and  $X = (x_1, \dots, x_{|V|})^t$ . Note that the same symbol  $x_i$  denotes both an element of the group  $\text{coker}(L(G))$  and a basis element of the free abelian group  $Z^{|V|}$ .

The finitely generated abelian group  $\text{coker}(L(G))$  can be described in terms of the Smith normal form (or simply SNF) of  $L(G)$ . Two integral matrices  $A$  and  $B$

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\*Supported by NSF of the People's Republic of China(Grant No. 10871189 and No. 10671191).

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of order  $n$  are equivalent (written by  $A \sim B$ ) if there are unimodular matrices  $P$  and  $Q$  such that  $B = PAQ$ . Equivalently,  $B$  is obtainable from  $A$  by a sequence of elementary row and column operations: (1) the interchange of two rows or columns, (2) the multiplication of any row or column by  $-1$ , (3) the addition of any integer times of one row (resp. column) to another row (resp. column). It is easy to see that  $A \sim B$  implies that  $\text{coker}(A) \cong \text{coker}(B)$ . Given any  $|V| \times |V|$  unimodular matrices  $P$  and  $Q$  and any integral matrix  $A$  with  $PAQ = \text{diag}(a_1, \dots, a_{|V|})$ , it is easy to see that  $Z^{|V|}/\text{im}(A) \cong (Z/a_1Z) \oplus \dots \oplus (Z/a_{|V|}Z)$ . Here, the rank of  $L(G)$  is  $|V| - 1$ , with kernel generated by the transpose of the vector  $(1, \dots, 1)$ . Thus we can assume the SNF of  $L(G)$  is  $\text{diag}(t_1, \dots, t_{|V|-1}, 0)$ , and it induces an isomorphism

$$\text{coker}(L(G)) \cong K(G) \oplus Z. \quad (1.2)$$

where  $K(G) = (Z/t_1Z) \oplus (Z/t_2Z) \oplus \dots \oplus (Z/t_{|V|-1}Z)$ .

In [1] and [5 (Chapter 14)], the finite abelian group  $K(G)$  is defined to be the critical group of  $G$ . Its invariant factors  $t_1, t_2, \dots, t_{|V|-1}$  can be computed in the following way: for  $1 \leq i < |V|$ ,  $t_i = \Delta_i / \Delta_{i-1}$  where  $\Delta_0 = 1$  and  $\Delta_i$  is the  $i$ -th determinantal divisor of  $L(G)$ , defined as the greatest common divisor of all  $i \times i$  minor subdeterminants of  $L(G)$ . From the well known Kirchhoff's Matrix-Tree Theorem [7, Theorem 13.2.1] we know that  $t_1 \cdots t_{|V|-1}$  equals the number  $\kappa$  of spanning trees of  $G$ . It follows that the invariant factors of  $K(G)$  can be used to distinguish pairs of non-isomorphic graphs which have the same  $\kappa$ , and so there is considerable interest in their properties. If  $G$  is a simple connected graph, the invariant factor  $t_1$  of  $K(G)$  must be equal to 1, however, most of them are not easy to be determined.

Compared to the number of the results on the spanning tree number  $\kappa$ , there are relatively few results describing the critical group structure of  $K(G)$  in terms of the structure of  $G$ . There are also very few interesting infinite family of graphs for which the group structure has been complete determined (see [2, 3, 4, 6, 7, 8], and the references therein). In this paper, we describe the critical group structure of Cartesian product graph  $C_4 \times C_n$  ( $n \geq 3$ ) completely, where  $C_n$  is the cycle on  $n$  vertices.

Given two disjoint graphs  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$ , their Cartesian product is the graph  $G_1 \times G_2$  whose vertex set is the cartesian product  $V_1 \times V_2$ . Suppose  $u_1, u_2 \in V_1$  and  $v_1, v_2 \in V_2$ . Then  $(u_1, v_1)$  is adjacent to  $(u_2, v_2)$  if and only if one of the following conditions satisfied: (i)  $u_1 = u_2$  and  $(v_1, v_2) \in E_2$ , or (ii)  $(u_1, u_2) \in E_1$  and  $v_1 = v_2$ . One may view  $G_1 \times G_2$  as the graph obtained from  $G_2$  by replacing each of its vertices with a copy of  $G_1$ , and each of its edges with  $|V_1|$  edges joining corresponding vertices of  $G_1$  in the two copies. From the definition of the Cartesian product of two graphs, it is easy to see that there are  $n$  layers of  $C_4 \times C_n$ , each of which is a copy of  $C_4$ . Let  $Z_n$  denote  $Z/nZ$ , then for  $i \in Z_n$ ,  $j \in Z_4$ , let

$v_j^i$  denote the  $j$ -th vertex in the  $i$ -th layer of  $C_4 \times C_n$ . The vertex  $v_j^i$  is adjacent to vertices  $v_j^l$  and  $v_k^i$ , where  $l = i \pm 1$ ,  $k = j \pm 1 \pmod{4}$  (see Fig. 1).

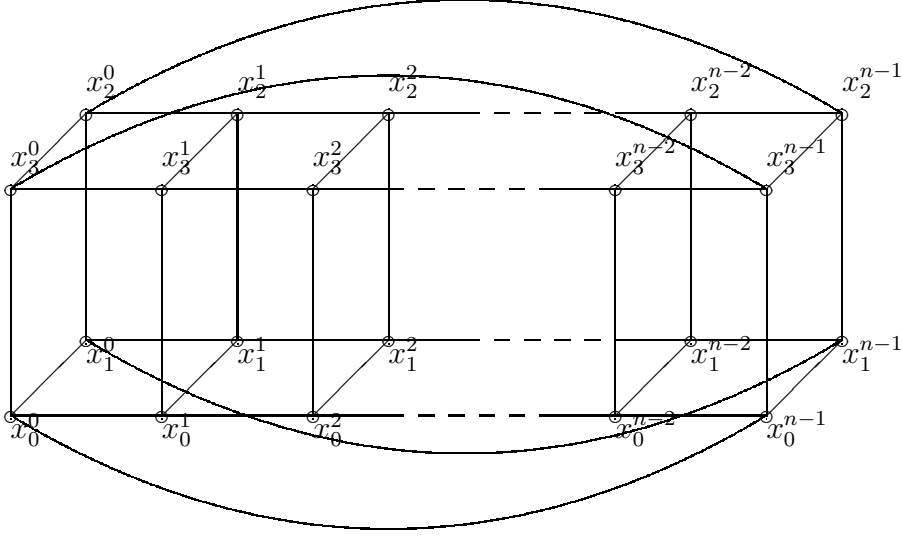


Fig. 1. Graph  $C_4 \times C_n$ .

## 2 Preliminaries

Let  $m$  be a positive integer. Denote  $\alpha(m) = \frac{m+2+\sqrt{m^2+4m}}{2}$ ,  $\beta(m) = \frac{m+2-\sqrt{m^2+4m}}{2}$ ,  $u_p(m) = \frac{1}{\alpha(m)-\beta(m)} (\alpha^p(m) - \beta^p(m))$ ,  $v_p(m) = \alpha^p(m) + \beta^p(m)$ , for  $p \in \mathbb{R}$ .

By the following proposition 2.1, it is easy to see that for every integer  $p \geq 0$ ,  $u_p(m)$  and  $v_p(m)$  are integral. The propositions 2.1 and 2.2 can be easily proved by induction.

**Proposition 2.1.** *If  $p$  is integral, then*

$$\begin{cases} u_p(m) = (m+2)u_{p-1}(m) - u_{p-2}(m), \\ v_p(m) = (m+2)v_{p-1}(m) - v_{p-2}(m), \end{cases} \quad (2.1)$$

with initial values

$$\begin{cases} u_0(m) = 0, & u_1(m) = 1, \\ v_0(m) = 2, & v_1(m) = m+2. \end{cases} \quad (2.2)$$

And if  $q \geq 0$  is another integer, then  $u_{pq}(m) = v_{p(q-1)}(m)u_p(m) + u_{p(q-2)}(m)$ .

**Proposition 2.2.** *If  $p$  is a nonnegative integer, then*

$$\bullet \quad u_p(m) \equiv p \pmod{m}, \quad v_p(m) \equiv 2 \pmod{m}; \quad (2.3)$$

$$\bullet \quad v_{2p}(m) = m(m+4)u_p^2(m) + 2; \quad (2.4)$$

$$\bullet \quad u_{pq}(m) = \begin{cases} V_q(m)u_p(m), & \text{if } q \text{ is even,} \\ V'_q(m)u_p(m), & \text{if } q \text{ is odd,} \end{cases} \quad (2.5)$$

where

$$V_q(m) = \sum_{0 < 2i \leq q} v_{p(q+1-2i)}(m), \quad V'_q(m) = \left( \sum_{0 < 2i \leq q+1} v_{p(q+1-2i)}(m) \right) - 1. \quad (2.6)$$

If  $n$  is a positive integer of the form  $p_1^{t_1} \cdots p_k^{t_k}$  where the  $p_i$ 's are distinct primes, then let  $T_{p_i}(n)$  denote  $t_i$ . Let  $e_n = u_n(2)$ ,  $f_n = u_n(4)$ .

**Proposition 2.3.** *Let  $T_2(n) = t_2$ ,  $T_3(n) = t_3$ , for  $n \geq 2$ . Then we have  $T_2(e_n) = \begin{cases} 0, & \text{if } t_2 = 0, \\ t_2 + 1, & \text{if } t_2 > 0; \end{cases}$   $T_2(f_n) = t_2$ ;  $T_3(e_n) = t_3$ ; and  $T_3(f_n) = \begin{cases} 0, & \text{if } t_2 = 0, \\ t_3 + 1, & \text{if } t_2 > 0. \end{cases}$*

*Proof.* Let  $n = 2^{t_2}q$ , where  $q$  is odd.

By (2.5),  $e_n = V'_q(2)e_{2^{t_2}}$  and  $f_n = V'_q(4)f_{2^{t_2}}$ . By (2.3),  $v_p(2)$  and  $v_p(4)$  are even for every  $p$  and then from (2.6) we have that  $V'_q(2)$  and  $V'_q(4)$  are odd. Thus  $T_2(e_n) = T_2(e_{2^{t_2}})$  and  $T_2(f_n) = T_2(f_{2^{t_2}})$ . If  $t_2 = 0$ , then  $T_2(e_{2^{t_2}}) = T_2(e_1) = 0$  and  $T_2(f_{2^{t_2}}) = T_2(f_1) = 0$ . Now we prove by induction on  $t_2 > 0$  that  $T_2(e_{2^{t_2}}) = t_2 + 1$  and  $T_2(f_{2^{t_2}}) = t_2$ . This is valid if  $t_2 = 1$ . Since from (2.4), (2.5) and (2.6) it follows that  $e_{2^{t_2}} = v_{2^{t_2-1}}(2)e_{2^{t_2-1}} = (12e_{2^{t_2-2}}^2 + 2)e_{2^{t_2-1}}$  and  $f_{2^{t_2}} = v_{2^{t_2-1}}(4)f_{2^{t_2-1}} = (32f_{2^{t_2-2}}^2 + 2)f_{2^{t_2-1}}$ , then by the induction hypothesis we have that  $T_2(e_{2^{t_2}}) = T_2(12e_{2^{t_2-2}}^2 + 2) + T_2(e_{2^{t_2-1}}) = 1 + t_2$  and  $T_2(f_{2^{t_2}}) = T_2(32f_{2^{t_2-2}}^2 + 2) + T_2(f_{2^{t_2-1}}) = 1 + t_2 - 1 = t_2$ . Thus  $T_2(e_n) = t_2 + 1$  and  $T_2(f_n) = t_2$ .

Let  $n = 3^{t_3}\gamma$ , where  $3 \nmid \gamma$ .

By (2.5),  $e_n = \begin{cases} V'_\gamma(2)e_{3^{t_3}}, & \text{if } 2 \nmid \gamma, \\ V_\gamma(2)e_{3^{t_3}}, & \text{if } 2 \mid \gamma. \end{cases}$  Note that  $v_n(2) = 4v_{n-1}(2) - v_{n-2}(2) \equiv v_{n-1}(2) - v_{n-2}(2) \equiv -v_{n-3} \pmod{3}$ ,  $v_0(2) = 2$ ,  $v_1(2) = 4 \equiv 1 \pmod{3}$ ,  $v_2(2) = 14 \equiv 2 \pmod{3}$ ,  $v_3(2) = 52 \equiv 1 \pmod{3}$ ,  $v_4(2) = 194 \equiv 2 \pmod{3}$ ,  $v_5(2) = 724 \equiv 1 \pmod{3}$ . Then it is not difficult to see that if  $2 \nmid \gamma$  then  $v_{3^{t_3}(\gamma+1-2i)}(2) \equiv 2 \pmod{3}$ ; if  $2 \mid \gamma$  then  $v_{3^{t_3}(\gamma+1-2i)}(2) \equiv 1 \pmod{3}$ . Hence, if  $2 \nmid \gamma$ , then  $V'_\gamma(2) \equiv 2 \times \frac{\gamma+1}{2} - 1 = \gamma \pmod{3}$ ; if  $2 \mid \gamma$ , then  $V_\gamma(2) \equiv \frac{\gamma}{2} \pmod{3}$ . It follows that neither  $V_\gamma(2)$  ( $\gamma$  is even) nor  $V'_\gamma(2)$  ( $\gamma$  is odd) contains the divisor 3, and hence  $T_3(e_n) = T_3(e_{3^{t_3}})$ . Now we prove by induction on  $t_3$  that  $T_3(e_{3^{t_3}}) = t_3$ . It is valid if  $t_3 = 0$ , or 1. Since from (2.4) and (2.5) we have that  $e_{3^{t_3}} = (v_{3^{t_3-1,2}}(2) + v_0(2) - 1)e_{3^{t_3-1}} = (12e_{3^{t_3-1}}^2 + 3)e_{3^{t_3-1}}$ . So by the induction hypothesis we have  $T_3(e_{3^{t_3}}) = T_3(12e_{3^{t_3-1}}^2 + 3) + T_3(e_{3^{t_3-1}}) = 1 + t_3 - 1 = t_3$ . Thus  $T_3(e_n) = t_3$ .

If  $t_2 = 0$ , namely  $n$  is odd, then we have  $f_n = 6f_{n-1} - f_{n-2} \equiv -f_{n-2} \equiv \cdots \equiv (-1)^{\frac{n-1}{2}}f_1 \pmod{3}$ . Note that  $f_1 = 1$ , so  $T_3(f_n) = 0$ .

If  $t_2 > 0$ , namely  $n$  is even, then we can write  $n = 3^{t_3} \cdot 2\epsilon$ , where  $3 \nmid \epsilon$ . By (2.5),  $f_n = \begin{cases} V'_\epsilon(4)f_{2 \cdot 3^{t_3}}, & \text{if } 2 \nmid \epsilon, \\ V_\epsilon(4)f_{2 \cdot 3^{t_3}}, & \text{if } 2 \mid \epsilon. \end{cases}$  By (2.1),  $v_n(4) = 6v_{n-1}(4) - v_{n-2}(4) \equiv -v_{n-2}(4) \equiv$

$\cdots \equiv (-1)^{\frac{n}{2}} v_0(4) = (-1)^{\frac{n}{2}} 2 \pmod{3}$ . Then from (2.6) we have that if  $2 \nmid \epsilon$ ,  $V'_\epsilon(4) \equiv 2 \times \frac{\epsilon+1}{2} - 1 = \epsilon \pmod{3}$ ; if  $2 \mid \epsilon$ , then  $V_\epsilon(4) \equiv (-2) \times \frac{\epsilon}{2} = -\epsilon \pmod{3}$ . Thus neither  $V'_\epsilon(4)$  ( $\epsilon$  is odd) nor  $V_\epsilon(4)$  ( $\epsilon$  is even) is divisible by 3. So  $T_3(f_n) = T_3(f_{2 \cdot 3^{t_3}})$ . Now we prove by induction on  $t_3$  that  $T_3(f_{2 \cdot 3^{t_3}}) = t_3 + 1$ . If  $t_3 = 0$  or 1, we have  $f_2 = 6$  and  $f_6 = 6930$  respectively, so it is valid. Since from (2.4), (2.5) and (2.6) it follows that  $f_{2 \cdot 3^{t_3}} = (v_{2 \cdot 3^{t_3-1} \cdot 2}(4) + v_0(4) - 1)f_{2 \cdot 3^{t_3-1}} = (32f_{2 \cdot 3^{t_3-1}}^2 + 3)f_{2 \cdot 3^{t_3-1}}$ , then by the induction hypothesis we have  $T_3(f_{2 \cdot 3^{t_3}}) = T_3(32f_{2 \cdot 3^{t_3-1}}^2 + 3) + T_3(f_{2 \cdot 3^{t_3-1}}) = 1 + t_3$ .  $\square$

### 3 System of relations for the cokernel of the Laplacian on $C_4 \times C_n$

Now we work on the system of relations of the cokernel of the Laplacian of  $C_4 \times C_n$ . Let  $x_j^i = (0, \dots, 0, 1, 0, \dots, 0) \in Z^{4n}$ , whose unique nonzero 1 is in the position corresponding to vertex  $v_j^i$ . It follows from the relations of  $\text{coker} L(C_4 \times C_n)$  that we can get the system of equations:

$$4x_j^i - (x_{j+1}^i + x_{j-1}^i) - x_j^{i+1} - x_j^{i-1} = 0, \quad j \in Z_4, \quad i \in Z_n. \quad (3.1)$$

**Lemma 3.1.** *There are three sequences of integral numbers  $(a_i)_{i \geq 0}$ ,  $(b_i)_{i \geq 0}$ ,  $(c_i)_{i \geq 0}$  such that*

$$x_j^i = a_i x_j^1 + b_i (x_{j+1}^1 + x_{j-1}^1) + c_i x_{j+2}^1 - a_{i-1} x_j^0 - b_{i-1} (x_{j+1}^0 + x_{j-1}^0) - c_{i-1} x_{j+2}^0, \quad (3.2)$$

where  $j \in Z_4$ ,  $1 \leq i \leq n$ . Moreover, the numbers in the above sequences have recurrence relations and initial conditions as follows

$$\begin{cases} a_i = \frac{1}{4}(i + u_i(4) + 2u_i(2)), & (i \geq 0), \\ b_i = \frac{1}{4}(i - u_i(4)), & (i \geq 0), \\ c_i = \frac{1}{4}(i + u_i(4) - 2u_i(2)), & (i \geq 0). \end{cases} \quad (3.3)$$

*Proof.* From (3.1), it follows that

$$x_j^{i+1} = 4x_j^i - (x_{j+1}^i + x_{j-1}^i) - x_j^{i-1}, \quad \text{for } j \in Z_4, \quad 2 \leq i \leq n-1. \quad (3.4)$$

This lemma is valid for the cases of  $i = 1, 2$ . Suppose that  $x_j^l = a_l x_j^1 + b_l (x_{j+1}^1 + x_{j-1}^1) + c_l x_{j+2}^1 - a_{l-1} x_j^0 - b_{l-1} (x_{j+1}^0 + x_{j-1}^0) - c_{l-1} x_{j+2}^0$ , for  $l \leq h$ . Then from the induction assumption and equation (3.4), it follows that  $x_j^{h+1} = 4x_j^h - (x_{j+1}^h + x_{j-1}^h) - x_j^{h-1} = 4(a_h x_j^1 + b_h (x_{j+1}^1 + x_{j-1}^1) + c_h x_{j+2}^1 - a_{h-1} x_j^0 - b_{h-1} (x_{j+1}^0 + x_{j-1}^0) - c_{h-1} x_{j+2}^0) - (a_h x_{j+1}^1 + b_h (x_{j+2}^1 + x_j^1) + c_h x_{j-1}^1 - a_{h-1} x_{j+1}^0 - b_{h-1} (x_{j+2}^0 + x_j^0) - c_{h-1} x_{j-1}^0) - (a_h x_{j-1}^1 + b_h (x_j^1 + x_{j-2}^1) + c_h x_{j+1}^1 - a_{h-1} x_{j-1}^0 - b_{h-1} (x_j^0 + x_{j-2}^0) - c_{h-1} x_{j+1}^0) - (a_{h-1} x_j^1 + b_{h-1} (x_{j+1}^1 +$

$$\begin{aligned}
& x_{j-1}^1) + c_{h-1}x_{j+2}^1 - a_{h-2}x_j^0 - b_{h-2}(x_{j+1}^0 + x_{j-1}^0) - c_{h-2}x_{j+2}^0) = (4a_h - 2b_h - a_{h-1})x_j^1 + \\
& (4b_h - a_h - c_h - b_{h-1})(x_{j+1}^1 + x_{j-1}^1) + (4c_h - 2b_h - c_{h-1})x_{j+2}^1 - (4a_{h-1} - 2b_{h-1} - \\
& a_{h-2})x_j^0 - (4b_{h-1} - a_{h-1} - c_{h-1} - b_{h-2})(x_{j+1}^0 + x_{j-1}^0) - (4c_{h-1} - 2b_{h-1} - c_{h-2})x_{j+2}^0 \\
& = a_{h+1}x_j^1 + b_{h+1}(x_{j+1}^1 + x_{j-1}^1) + c_{h+1}x_{j+2}^1 - a_hx_j^0 - b_h(x_{j+1}^0 + x_{j-1}^0) - c_hx_{j+2}^0.
\end{aligned}$$

Thus (3.2) holds by induction.

From the process of induction just now, it is easy to see that

$$\begin{cases} a_{i+1} = 4a_i - 2b_i - a_{i-1}, \\ b_{i+1} = 4b_i - (a_i + c_i) - b_{i-1}, \\ c_{i+1} = 4c_i - 2b_i - c_{i-1}, \end{cases} \quad (3.5)$$

for  $i \geq 1$ . Let  $\tau_i = a_i + c_i$  and  $\eta_i = a_i - c_i$ . After a short calculation, we can get

$$\begin{cases} \eta_{i+1} = 4\eta_i - \eta_{i-1}, \\ \eta_0 = 0, \quad \eta_1 = 1; \\ \tau_{i+2} - 8\tau_{i+1} + 14\tau_i - 8\tau_{i-1} + \tau_{i-2} = 0, \\ \tau_0 = 0, \quad \tau_1 = 1. \end{cases}$$

By proposition 2.1, we have  $\eta_i = u_i(2) = e_i$ . Let  $\phi_i = 2\tau_i - i$ , then one can verify that  $\phi_{i+2} = 6\phi_{i+1} - \phi_i$ , with  $\phi_0 = 0$  and  $\phi_1 = 1$ . Immediately,  $\phi_i = u_i(4) = f_i$ , and then  $\tau_i = \frac{1}{2}(i + u_i(4))$ . Now the equalities in (3.3) can be verified directly.  $\square$

We know from lemma 3.1 that the system of equation (3.2) has at most 8 generators, i.e., each  $x_j^i$  can be expressed in terms of  $x_0^0, x_1^0, x_2^0, x_3^0, x_0^1, x_1^1, x_2^1, x_3^1$ . So there are at least  $4n - 8$  diagonal entries of the Smith normal form of  $L(G)$  are equal to 1, however the remaining invariant factors of  $\text{coker}(C_4 \times C_n)$  hide inside the relations matrix induced by  $x_0^0, x_1^0, x_2^0, x_3^0, x_0^1, x_1^1, x_2^1, x_3^1$ .

$$\text{Let } Y = (x_0^1, x_1^1, x_2^1, x_3^1, x_0^0, x_1^0, x_2^0, x_3^0)^t, A_n = \begin{pmatrix} a_n & b_n & c_n & b_n \\ b_n & a_n & b_n & c_n \\ c_n & b_n & a_n & b_n \\ b_n & c_n & b_n & a_n \end{pmatrix} \text{ and}$$

$$M = \begin{pmatrix} A_{n+1} & -A_n \\ A_n & -A_{n-1} \end{pmatrix}. \text{ From (3.2) and the cyclic structure of } C_4 \times C_n, \text{ we have}$$

$$\begin{cases} x_j^0 = x_j^n = a_nx_j^1 + b_n(x_{j+1}^1 + x_{j-1}^1) + c_nx_{j+2}^1 - a_{n-1}x_j^0 \\ \quad - b_{n-1}(x_{j+1}^0 + x_{j-1}^0) - c_{n-1}x_{j+2}^0, \\ x_j^1 = x_j^{n+1} = a_{n+1}x_j^1 + b_{n+1}(x_{j+1}^1 + x_{j-1}^1) + c_{n+1}x_{j+2}^1 - a_nx_j^0 \\ \quad - b_n(x_{j+1}^0 + x_{j-1}^0) - c_nx_{j+2}^0, \end{cases}$$

where  $0 \leq j \leq 3$ . Therefore

$$(M - I)Y = 0. \quad (3.6)$$

From the argument above, we know that one can reduce  $L(G)$  to  $I_{4n-8} \oplus (M - I)$  by performing some row and column operations up to equivalence. Now we only need to evaluate the SNF of  $M - I$ .

## 4 Analysis of the coefficients of the Smith normal form of $M - I$

If we multiply the last 4 rows of  $M - I$  by  $-1$ , then we have that

$$\begin{pmatrix} A_{n+1} - I_4 & -A_n \\ A_n & -A_{n-1} - I_4 \end{pmatrix} \sim \begin{pmatrix} A_{n+1} - I_4 & -A_n \\ -A_n & A_{n-1} + I_4 \end{pmatrix}. \quad (4.1)$$

From lemma 3.1, one can verify that  $a_{i+1} + c_{i+1} + 2b_{i+1} = a_i + c_i + 2b_i + 1$ , for each  $i \in N$ , and it results that each line sum of the right matrix of (4.1) is equal to 0. Immediately, we have the following lemma.

**Lemma 4.1.**  $M - I \sim (0) \oplus M_1$ , where  $M_1$  is the submatrix of  $M - I$  resulting from the deletion of the first row and column.

Let  $h_n = e_n + e_{n+1}$ ,  $g_n = f_n + f_{n+1}$ ,  $p_i = e_i + e_{n-i}$ ,  $q_i = f_i + f_{n-i}$ , and let

$$L_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \quad R_1 = \begin{pmatrix} -1 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then one can check that  $L_1$  and  $R_1$  are unimodular matrices and

$$L_1 M_1 R_1 = \begin{pmatrix} 0 & 0 & 0 & n & n & 0 & 0 \\ 0 & p_{-1} & p_0 & 0 & 0 & 0 & 0 \\ 0 & p_0 & p_1 & 0 & 0 & 0 & 0 \\ \frac{q_{-1}+q_0}{2} & \frac{p_{-1}+q_{-1}}{2} & \frac{p_0+q_0}{2} & \frac{n-q_{-1}}{4} & \frac{n-q_0}{4} & 0 & 0 \\ \frac{q_0+q_1}{2} & \frac{p_0+q_0}{2} & \frac{p_1+q_1}{2} & \frac{n-q_0}{4} & \frac{n-q_1}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{n+p_{-1}}{2} & \frac{n+p_0}{2} & p_{-1} & p_0 \\ 0 & 0 & 0 & \frac{n+p_0}{2} & \frac{n+p_1}{2} & p_0 & p_1 \end{pmatrix}.$$

Putting  $m = 2$  and 4, then it follows from proposition 2.1 that

$$\begin{cases} p_{i+1} = 4p_i - p_{i-1}, \\ q_{i+1} = 6q_i - q_{i-1}. \end{cases} \quad (4.2)$$

Let  $M_2 = L_1 M_1 R_1$  and  $U = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & -1 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & -1 & 4 \end{pmatrix}.$

Then by (4.2) we have

$$U^i M_2 = \begin{pmatrix} 0 & 0 & 0 & n & n & 0 & 0 \\ 0 & p_{i-1} & p_i & 0 & 0 & 0 & 0 \\ 0 & p_i & p_{i-1} & 0 & 0 & 0 & 0 \\ \frac{q_{i-1}+q_i}{2} & \frac{p_{i-1}+q_{i-1}}{2} & \frac{p_i+q_i}{2} & \frac{n-q_{i-1}}{4} & \frac{n-q_i}{4} & 0 & 0 \\ \frac{q_i+q_{i+1}}{2} & \frac{p_i+q_i}{2} & \frac{p_{i+1}+q_{i+1}}{2} & \frac{n-q_i}{4} & \frac{n-q_{i+1}}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{n+p_{i-1}}{2} & \frac{n+p_i}{2} & p_{i-1} & p_i \\ 0 & 0 & 0 & \frac{n+p_0}{2} & \frac{n+p_{i+1}}{2} & p_i & p_{i+1} \end{pmatrix}. \quad (4.3)$$

Now we distinguish two cases.

**Case 1**  $n = 2s + 1$  **odd.**

In this case, by (4.2) one can verify that

$$\begin{pmatrix} p_s & p_{s+1} \\ p_{s+1} & p_{s+2} \end{pmatrix} = \begin{pmatrix} h_s & h_s \\ h_s & 3h_s \end{pmatrix}, \quad \begin{pmatrix} q_s & q_{s+1} \\ q_{s+1} & q_{s+2} \end{pmatrix} \begin{pmatrix} g_s & g_s \\ g_s & 5g_s \end{pmatrix}. \quad (4.4)$$

Let  $i = s + 1$  in (4.3), then by (4.4) we have

$$U^{s+1} M_2 = \begin{pmatrix} 0 & 0 & 0 & n & n & 0 & 0 \\ 0 & h_s & h_s & 0 & 0 & 0 & 0 \\ 0 & h_s & 3h_s & 0 & 0 & 0 & 0 \\ g_s & \frac{g_s+h_s}{2} & \frac{g_s+h_s}{2} & \frac{n-g_s}{4} & \frac{n-g_s}{4} & 0 & 0 \\ 3g_s & \frac{g_s+h_s}{2} & \frac{5g_s+3h_s}{2} & \frac{n-g_s}{4} & \frac{n-g_s}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{n+h_s}{2} & \frac{n+h_s}{2} & h_s & h_s \\ 0 & 0 & 0 & \frac{n+h_s}{2} & \frac{n+3h_s}{2} & h_s & 3h_s \end{pmatrix}.$$

Let

$$L_2 = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 1 & 0 & 0 & -2 \\ -1 & 2 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & -1 & -1 & 0 & 0 & 0 & -1 \end{pmatrix}.$$



It is clear that  $L_2$  and  $R_2$  are unimodular matrices. By a direct calculation, we get

$$L_2 U^{s+1} M_2 R_2 = X \oplus Y, \quad (4.5)$$

$$\text{where } X = \begin{pmatrix} 0 & 2h_s & 0 \\ h_s & 0 & 2h_s \\ g_s & h_s & 0 \end{pmatrix} \text{ and } Y = \begin{pmatrix} n & 0 & 0 & 0 \\ 0 & h_s & 0 & 0 \\ \frac{n+h_s}{2} & 0 & h_s & 0 \\ \frac{n-g_s}{4} & \frac{h_s+g_s}{2} & 0 & g_s \end{pmatrix}.$$

Using the standard method for calculating the determinant factors we have that

$$\text{SNF}(X) = \text{diag} \left( (h_s, g_s), h_s, \frac{4h_s g_s}{(h_s, g_s)} \right)$$

and

$$\text{SNF}(Y) = \text{diag} \left( (n, h_s, g_s), \frac{(n, h_s)(h_s, g_s)}{(n, h_s, g_s)}, \frac{h_s(nh_s, ng_s, h_s g_s)}{(n, h_s)(h_s, g_s)}, \frac{nh_s g_s}{(nh_s, ng_s, h_s g_s)} \right).$$

From above, now it is easy to see that in this case  $\text{SNF}(M_1) = \text{SNF}(M_2) =$

$$\text{diag} \left( (n, h_s, g_s), (h_s, g_s), \frac{(n, h_s)(h_s, g_s)}{(n, h_s, g_s)}, h_s, \frac{h_s(nh_s, ng_s, h_s g_s)}{(n, h_s)(h_s, g_s)}, \frac{h_s g_s}{(h_s, g_s)}, \frac{4nh_s g_s}{(nh_s, ng_s, h_s g_s)} \right).$$

**Case 2  $n = 2s$  even.**

In this case, by (4.2) one can verify that

$$\begin{pmatrix} p_s & p_{s+1} \\ p_{s+1} & p_{s+2} \end{pmatrix} = \begin{pmatrix} 2e_s & 4e_s \\ 4e_s & 14e_s \end{pmatrix}, \quad \begin{pmatrix} q_s & q_{s+1} \\ q_{s+1} & q_{s+2} \end{pmatrix} = \begin{pmatrix} 2f_s & 6f_s \\ 6f_s & 34f_s \end{pmatrix}.$$

Apply (4.3), we have

$$U^{s+1} M_2 = \begin{pmatrix} 0 & 0 & 0 & 2s & 2s & 0 & 0 \\ 0 & 2e_s & 4e_s & 0 & 0 & 0 & 0 \\ 0 & 4e_s & 14e_s & 0 & 0 & 0 & 0 \\ 4f_s & f_s + e_s & 3f_s + 2e_s & \frac{s-f_s}{2} & \frac{s-3f_s}{2} & 0 & 0 \\ 20f_s & 3f_s + 2e_s & 17f_s + 7e_s & \frac{s-3f_s}{2} & \frac{s-17f_s}{2} & 0 & 0 \\ 0 & 0 & 0 & s + e_s & s + 2e_s & 2e_s & 4e_s \\ 0 & 0 & 0 & s + 2e_s & s + 7e_s & 4e_s & 14e_s \end{pmatrix}.$$

Let

$$L_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & -1 \\ -1 & -4 & 1 & 7 & -1 & 0 & 0 \\ 0 & 5 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & -2 & 1 & 1 & 0 & 0 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 0 & -2 & 0 & 1 & -2 & 0 & 0 \\ 0 & 6 & 0 & 0 & 5 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 2 & 6 & 0 & -4 & 6 & 1 & 2 \\ -1 & -6 & 0 & 4 & -1 & -1 & -2 \\ 0 & -3 & 2 & 2 & -3 & 1 & -1 \\ 0 & 3 & -1 & -2 & 3 & 0 & 1 \end{pmatrix}.$$

Then we have

$$L_3 U^{s+1} M_2 R_3 = \begin{pmatrix} 2s & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2e_s & 0 & 0 & 0 & 0 & 0 \\ 3e_s & 0 & 6e_s & 0 & 0 & 0 & 0 \\ s - 2f_s & e_s + 4f_s & 0 & 8f_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6e_s & 0 & 0 \\ s & 0 & 0 & 0 & 0 & e_s & 0 \\ \frac{1}{2}(f_s + s) & f_s & 0 & 0 & 3e_s & f_s & 2f_s \end{pmatrix}. \quad (4.6)$$

Let  $M_3$  denote the matrix on the right side of (4.6). If we can further reduce  $M_3$  to the direct product of some small matrices as in the above case of  $n$  being odd, then the calculation will become easier. Unfortunately, we can not achieve it.

Let

$$M'_3 = \begin{pmatrix} s & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e_s & 0 & 0 & 0 & 0 & 0 \\ 3e_s & 0 & 3e_s & 0 & 0 & 0 & 0 \\ s - 2f_s & e_s + 4f_s & 0 & f_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3e_s & 0 & 0 \\ s & 0 & 0 & 0 & 0 & e_s & 0 \\ \frac{1}{2}(f_s + s) & f_s & 0 & 0 & 3e_s & f_s & f_s \end{pmatrix},$$

$$L_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad R_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & -4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

It is clear that  $L_4$  and  $R_4$  are unimodular matrices and  $L_4 M'_3 R_4 = E \oplus F$ , where

$$E = \begin{pmatrix} s & 0 & 0 & 0 \\ \frac{1}{2}(s + f_s) & f_s & 0 & 0 \\ 0 & 0 & e_s & 0 \\ 0 & 0 & 0 & 3e_s \end{pmatrix}, \quad F = \begin{pmatrix} f_s & 0 & 0 \\ 0 & e_s & 0 \\ 0 & 0 & 3e_s \end{pmatrix}. \quad (4.7)$$

Now we can compute the determinantal divisors of  $E$  and  $F$  and furthermore obtain the SNF of  $M'_3$ . Here we directly give the result and omit the details of computation. However we must say that proposition 2.3 plays an important role in this computation.

$$\text{SNF}(E) = \begin{cases} \text{diag} \left( (s, e_s, f_s), \frac{(s, e_s)(e_s, f_s)}{(s, e_s)(f_s)}, \frac{e_s(se_s, sf_s, e_s f_s)}{(s, e_s)(e_s, f_s)}, \frac{3se_s f_s}{(se_s, sf_s, e_s f_s)} \right), & \text{if } 2 \nmid s, \\ \text{diag} \left( (s, e_s, f_s), \frac{(s, e_s)(e_s, f_s)}{(s, e_s)(f_s)}, \frac{3e_s(se_s, sf_s, e_s f_s)}{(s, e_s)(e_s, f_s)}, \frac{se_s f_s}{(se_s, sf_s, e_s f_s)} \right), & \text{if } 2 \mid s; \end{cases}$$

and

$$\text{SNF}(F) = \begin{cases} \text{diag} \left( (e_s, f_s), e_s, \frac{3e_s f_s}{(e_s, f_s)} \right), & \text{if } 2 \nmid s, \\ \text{diag} \left( (e_s, f_s), 3e_s, \frac{e_s f_s}{(e_s, f_s)} \right), & \text{if } 2 \mid s. \end{cases}$$

Then it is not hard to see that  $\text{SNF}(M'_3) =$

$$\begin{cases} \text{diag} \left( (s, e_s, f_s), (e_s, f_s), \frac{(s, e_s)(e_s, f_s)}{(s, e_s, f_s)}, e_s, \frac{e_s(se_s, ef_s, e_s f_s)}{(s, e_s)(e_s, f_s)}, \frac{3e_s f_s}{(e_s, f_s)}, \frac{3se_s f_s}{(se_s, ef_s, e_s f_s)} \right), & \text{if } 2 \nmid s, \\ \text{diag} \left( (s, e_s, f_s), (e_s, f_s), \frac{(s, e_s)(e_s, f_s)}{(s, e_s, f_s)}, 3e_s, \frac{3e_s(se_s, sf_s, e_s f_s)}{(s, e_s)(e_s, f_s)}, \frac{e_s f_s}{(e_s, f_s)}, \frac{se_s f_s}{(se_s, ef_s, e_s f_s)} \right), & \text{if } 2 \mid s. \end{cases}$$

Note that  $M_3$  is obtained from  $M'_3$  by multiplying its rows 1, 2, 5 by 2, columns 3, 7 by 2, column 4 by 8. Then we have that there are integers  $t_i$  such that  $S_i(M_3) = 2^{t_i} S_i(M'_3)$ , for  $1 \leq i \leq 7$ .

- $n = 2s$  with  $s$  odd.

It follows from proposition 2.3 that  $2 \nmid e_s$  and  $2 \nmid f_s$ . Moreover,  $\Delta_i(M'_3)$  is odd and hence  $S_i(M'_3)$  is odd. Since  $\det(M_3[3, 4, 6, 7|1, 2, 5, 6]) = -9e_s^3(e_s + 4f_s)$  is odd, where  $M_3[3, 4, 6, 7|1, 2, 5, 6]$  is the submatrix that lies in the rows 3, 4, 6, 7 and columns 1, 2, 5, 6 of  $M_3$ . Thus  $t_1 = t_2 = t_3 = t_4 = 0$ . Note that every nonzero element in rows 1, 2, 5, columns 3, 4, 7 of  $M_3$  is even and on the main diagonal, so every  $5 \times 5$  submatrix of  $M_3$  must contain at least one row and at least one column of them. Thus  $2^2 \mid \Delta_5(M_3)$ . Since  $\det(M_3[1, 3, 4, 6, 7|1, 2, 3, 5, 6]) = 36se_s^3(e_s + 4f_s)$ , then  $2^3$  is not its divisor. Thus  $t_5 = 2$ . As above,  $2^4 \mid \Delta_6(M_3)$ , but  $\det(M_3[1, 3, 4, 5, 6, 7|1, 2, 3, 5, 6, 7]) = -144se_s^3f_s(e_s + 4f_s)$ , which is not divisible by  $2^5$ . So  $t_6 = 4 - 2 = 2$ . Finally, it is easy to see that  $t_7 = 8 - 4 = 4$ . Thus the SNF of  $M_3$  here is  $\text{diag} \left( (s, e_s, f_s), (e_s, f_s), \frac{(s, e_s)(e_s, f_s)}{(s, e_s, f_s)}, e_s, \frac{4e_s(se_s, sf_s, e_s f_s)}{(s, e_s)(e_s, f_s)}, \frac{12e_s f_s}{(e_s, f_s)}, \frac{48se_s f_s}{(se_s, sf_s, e_s f_s)} \right)$ .

- $n = 2s$  with  $s$  even.

Let  $t = T_2(s)$ , then from proposition 2.3, it follows that  $T_2(e_s) = t + 1$  and  $T_2(f_s) = t$ . It is clear that  $S_1(M_3) = S_1(M'_3)$ , so  $t_1 = 0$ . Since  $T_2(\det(M_3[6, 7|1, 2])) = T_2(sf_s) = 2t = T_2(\Delta_2(M'_3)) = 2t$ , then clearly  $t_2 = 0$ . It is not hard to see that the maximal power of 2 contained in each of the  $3 \times 3$  minor subdeterminants of  $M_3$  is at least  $3t + 2$ , and then we can conclude that  $T_2(\Delta_3(M_3)) = 3t + 2$ , since  $\det(M_3[4, 6, 7|1, 2, 7]) = -2sf_s(e_s + 4f_s)$  is not divisible by  $2^{3t+3}$ . Then  $T_2(S_3(M_3)) = T_2(\Delta_3(M_3)) - T_2(\Delta_2(M_3)) = (3t + 2) - 2t = t + 2$ . So  $t_3 = T_2(S_3(M_3)) - T_2(S_3(M'_3)) = (t + 2) - t = 2$ . All the  $4 \times 4$  minor subdeterminants of  $M_3$  contain the divisor  $2^{4t+4}$ , and then we can say that  $T_2(\Delta_4(M_3)) = 4t + 4$ , since  $\det(M_3[3, 4, 6, 7|1, 2, 3, 7]) = -12se_s f_s(e_s + 4f_s)$  is not divisible by  $2^{4t+5}$ . Then  $T_2(S_4(M_3)) = T_2(\Delta_4(M_3)) - T_2(\Delta_3(M_3)) = (4t + 4) - (3t + 2) = t + 2$ . So  $t_4 = T_2(S_4(M_3)) - T_2(S_4(M'_3)) = (t + 2) - (t + 1) = 1$ . Go on in this way, we obtain that  $t_5 = t_6 = 1$  and  $t_7 = 3$ . Thus we get that SNF of  $M_3$  here is  $\text{diag} \left( (s, e_s, f_s), (e_s, f_s), \frac{4(s, e_s)(e_s, f_s)}{(s, e_s, f_s)}, 6e_s, \frac{6e_s(se_s, sf_s, e_s f_s)}{(s, e_s)(e_s, f_s)}, \frac{2e_s f_s}{(e_s, f_s)}, \frac{8se_s f_s}{(se_s, sf_s, e_s f_s)} \right)$ .

## 5 Conclusion

Now we can give the main result as follows.

**Theorem 5.1.** *If  $n = 2s + 1$  odd, then the critical group of  $C_4 \times C_n$  ( $n \geq 3$ ) is*

$$Z_{(n, h_s, g_s)} \oplus Z_{(h_s, g_s)} \oplus Z_{\frac{(n, h_s)(h_s, g_s)}{(n, h_s, g_s)}} \oplus Z_{h_s} \oplus Z_{\frac{h_s(nh_s, ng_s, h_s g_s)}{(n, h_s)(h_s, g_s)}} \oplus Z_{\frac{h_s g_s}{(h_s, g_s)}} \oplus Z_{\frac{4nh_s g_s}{(nh_s, ng_s, h_s g_s)}}.$$

*If  $n = 2s$  with  $s$  odd, then the critical group of  $C_4 \times C_n$  ( $n \geq 3$ ) is*

$$Z_{(s, e_s, f_s)} \oplus Z_{(e_s, f_s)} \oplus Z_{\frac{(s, e_s)(e_s, f_s)}{(s, e_s, f_s)}} \oplus Z_{e_s} \oplus Z_{\frac{4e_s(se_s, sf_s, e_s f_s)}{(s, e_s)(e_s, f_s)}} \oplus Z_{\frac{12e_s f_s}{(e_s, f_s)}} \oplus Z_{\frac{48se_s f_s}{(se_s, sf_s, e_s f_s)}}.$$

*If  $n = 2s$  with  $s$  even, then the critical group of  $C_4 \times C_n$  ( $n \geq 3$ ) is*

$$Z_{(s, e_s, f_s)} \oplus Z_{(e_s, f_s)} \oplus Z_{\frac{4(s, e_s)(e_s, f_s)}{(s, e_s, f_s)}} \oplus Z_{6e_s} \oplus Z_{\frac{6e_s(se_s, sf_s, e_s f_s)}{(s, e_s)(e_s, f_s)}} \oplus Z_{\frac{2e_s f_s}{(e_s, f_s)}} \oplus Z_{\frac{8se_s f_s}{(se_s, sf_s, e_s f_s)}}.$$

**Example 5.1.** *To give an illustration of theorem 5.1, we consider the three graphs  $C_4 \times C_4$ ,  $C_4 \times C_5$  and  $C_4 \times C_6$ . Note that  $e_0 = 0$ ,  $e_1 = 1$ ,  $e_2 = 4$ ,  $e_3 = 15$ ,  $e_4 = 56$ ,  $e_5 = 209$ ,  $e_6 = 780$ ,  $f_0 = 0$ ,  $f_1 = 1$ ,  $f_2 = 6$ ,  $f_3 = 35$ ,  $f_4 = 204$ ,  $f_5 = 1189$ ,  $f_6 = 6930$ . Then by theorem 5.1 we have that  $K(C_4 \times C_4) = (Z_2)^2 \oplus Z_8 \oplus (Z_{24})^3 \oplus Z_{96}$ ;  $K(C_4 \times C_5) = (Z_{19})^2 \oplus Z_{779} \oplus Z_{15580}$  and  $K(C_4 \times C_6) = Z_5 \oplus (Z_{15})^2 \oplus Z_{60} \oplus Z_{1260} \oplus Z_{5040}$ . Maple gives the identical result.*

Let  $H_n(m) = u_n(m) + u_{n+1}(m)$ . Clearly,  $H_n(2) = h_n$  and  $H_n(4) = g_n$ .

**Theorem 5.2.** *If  $n_1 \mid n_2$ , then  $K(C_4 \times C_{n_1})$  is a subgroup of  $K(C_4 \times C_{n_2})$ .*

*Proof.* We only need to prove that every invariant factor of  $K(C_4 \times C_{n_1})$  is a divisor of the corresponding one of  $K(C_4 \times C_{n_2})$ . We distinguish three cases.

*Case 1.*  $n_1 = 2s + 1$  and  $n_2 = (2k + 1)(2s + 1)$ .

Let  $p = 2s + 1$ ,  $q = 2k + 1$ , then  $H_{\lfloor \frac{n_2}{2} \rfloor}(m) = H_{pk+s}(m)$ . Since  $\alpha\beta = 1$ , then from the definition we can directly verify that  $u_{pk+s}(m) = v_{pk}u_s(m) + u_{pk-s}(m)$ ,  $u_{pk+s+1}(m) = v_{pk}(m)u_{s+1}(m) + u_{pk-s-1}(m)$ . Thus  $H_{pk+s}(m) = v_{pk}(m)H_s(m) + H_{pk-s-1}(m) = v_{pk}(m)H_s(m) + H_{p(k-1)+s} = \cdots = \left( \sum_{i=1}^k v_{ip}(m) + 1 \right) H_s(m)$ . It means that  $H_s(m) \mid H_{pk+s}(m)$  and hence  $h_s \mid h_{pk+s}$ ,  $g_s \mid g_{pk+s}$ . So every invariant factor of  $K(C_4 \times C_{2s+1})$  is a divisor of the corresponding one of  $K(C_4 \times C_{(2k+1)(2s+1)})$ .

*Case 2.*  $n_1 = 2s + 1$  and  $n_2 = 2k(2s + 1)$ .

Since one can verify that  $(u_n(m) + u_{n+1}(m))(u_n(m) - u_{n+1}(m)) = -u_{2n+1}(m)$  and  $u_n(m) = v_p(m)u_{n-p}(m) - u_{n-2p}(m)$ , we have that  $H_n(m) \mid u_{2n+1}(m)$  and if  $p \mid n$ , then  $u_p(m) \mid u_n(m)$ . Thus  $H_s(m) \mid u_{n_1}(m)$ , and  $u_{n_1}(m) \mid u_{kn_1}(m)$ . Then

$H_s(m) \mid u_{kn_1}(m)$ . It means that  $h_s \mid e_{kn_1}$  and  $g_s \mid f_{kn_1}$ . So every invariant factor of  $K(C_4 \times C_{2s+1})$  is a divisor of the corresponding one of  $K(C_4 \times C_{2k(2s+1)})$ .

*Case 3.*  $n_1 = 2s$  and  $n_2 = 2ks$ .

Since  $u_s(m) \mid u_{ks}(m)$ , then  $e_s \mid e_{ks}$  and  $f_s \mid f_{ks}$ . So every invariant factor of  $K(C_4 \times C_{2s})$  is a divisor of the corresponding one of  $K(C_4 \times C_{2ks})$ .  $\square$

**Theorem 5.3.** *The spanning tree number of  $C_4 \times C_n$  ( $n \geq 3$ ) is  $2^7 3^2 n e^{\frac{4}{2}} f_{\frac{n}{2}}^2$ , i.e.,*

$$\frac{n}{4^{n+1}} \left( (\sqrt{3} + 1)^n - (\sqrt{3} - 1)^n \right)^4 \cdot \left( (\sqrt{2} + 1)^n - (\sqrt{2} - 1)^n \right)^2.$$

*Proof.* We prove this theorem by distinguishing two cases.

*Case 1:*  $n = 2s + 1$ .

A direct calculation shows that  $h_s^4 = (e_s + e_{s+1})^4 = \frac{1}{4^{n+1}} \left( (\sqrt{3} + 1)^n - (\sqrt{3} - 1)^n \right)^4$  and  $g_s^2 = (f_s + f_{s+1})^2 = \frac{1}{4} \left( (\sqrt{2} + 1)^n - (\sqrt{2} - 1)^n \right)^2$ .

From (4.3), we know that the spanning tree number of  $C_4 \times C_n$  of this case is  $(\det X) \cdot (\det Y) = 4n h_s^4 g_s^2 = \frac{n}{4^{n+1}} \left( (\sqrt{3} + 1)^n - (\sqrt{3} - 1)^n \right)^4 \cdot \left( (\sqrt{2} + 1)^n - (\sqrt{2} - 1)^n \right)^2 = 2^7 3^2 n e^{\frac{4}{2}} f_{\frac{n}{2}}^2$ .

*Case 2:*  $n = 2s$ .

From (4.4), we know that the spanning tree number of  $C_4 \times C_n$  of this case is  $\det(M_3) = 2^8 3^2 s e_s^4 f_s^2 = 2^7 3^2 n e^{\frac{4}{2}} f_{\frac{n}{2}}^2$ .  $\square$

**Corollary 5.1.** *For every  $n \geq 3$ , we have that  $\prod_{j=1}^{n-1} \left( 4 - 2 \cos \frac{2\pi j}{n} \right)^2 \left( 6 - 2 \cos \frac{2\pi j}{n} \right) = \frac{1}{4^{n+2}} \left( (\sqrt{3} + 1)^n - (\sqrt{3} - 1)^n \right)^4 \cdot \left( (\sqrt{2} + 1)^n - (\sqrt{2} - 1)^n \right)^2$ .*

*Proof.* It is not difficult to know that the Laplacian eigenvalues of  $C_n$  are  $(2 - 2 \cos \frac{2\pi j}{n})$ ,  $0 \leq j \leq n-1$ . Then it follows from the argument of the second section of [7] that the Laplacian eigenvalues of  $C_4 \times C_n$  are:  $0, 2, 2, 4, 2 - 2 \cos \frac{2\pi j}{n}, 4 - 2 \cos \frac{2\pi j}{n}$  (with multiplicity 2),  $6 - 2 \cos \frac{2\pi j}{n}$ , where  $1 \leq j \leq n-1$ . Then by the well known Kirchhoff Matrix-Tree Theorem we know the spanning tree number of  $C_4 \times C_n$  is  $\frac{4}{n} \prod_{j=1}^{n-1} (2 - 2 \cos \frac{2\pi j}{n}) (4 - 2 \cos \frac{2\pi j}{n})^2 (6 - 2 \cos \frac{2\pi j}{n})$ . Since  $C_n$  has

$n$  spanning trees, we have  $\frac{1}{n} \prod_{j=1}^{n-1} (2 - 2 \cos \frac{2\pi j}{n}) = n$ . Thus the spanning tree number of  $C_4 \times C_n$  equals  $4n \prod_{j=1}^{n-1} (4 - 2 \cos \frac{2\pi j}{n})^2 (6 - 2 \cos \frac{2\pi j}{n})$ . Recall theorem 5.3, we

have that  $4n \prod_{j=1}^{n-1} (4 - 2 \cos \frac{2\pi j}{n})^2 (6 - 2 \cos \frac{2\pi j}{n}) = \frac{n}{4^{n+1}} \left( (\sqrt{3} + 1)^n - (\sqrt{3} - 1)^n \right)^4 \cdot \left( (\sqrt{2} + 1)^n - (\sqrt{2} - 1)^n \right)^2$ . So this corollary holds.  $\square$

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